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UNCERTAINTIES IN LCA

On the "rigorous proof of fuzzy error propagation with matrix-based LCI"

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Abstract
Purpose This short note re-examines the proof of fuzzy error propagation with matrix-based LCI in Heijungs and Tan (*Int J Life Cycle Assess* 15:1014–1019, 2010) paper (referred to as HT hereafter), published in this journal. We provide counter examples to the claims made therein, point out the key error in their proof and identify correct sufficient conditions under which the largest (smallest) values of the given α-cuts of fuzzy numbers in the technology matrix yield the smallest (largest) scaling factors.
Methods HT uses iterative perturbations of a matrix to seemingly provide a rigorous proof of this result. Flaws in their arguments are identified and demonstrated by way of a counterexample. A classical result on monotonic property of the inverse of M-matrices leads to the correct sufficient conditions under which HT result holds.
Results Since counter examples can be found, the result stated in HT is not, in general, guaranteed.
Conclusions As claimed in the HT paper, checking the upper and lower bounds of α-cuts may not be sufficient to describe the uncertainty (the full range of values) in the final inventory. However, slightly stronger conditions on the fuzzy technology matrix provide these inventory bounds.

Keywords α-Cuts · Fuzzy technology matrix · LCI · M-matrices

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1 Introduction

A recent article by Heijungs and Tan (2010), (referred to as HT hereafter), published in this journal purports to prove that the minimum (maximum) values of a final inventory can found by *simultaneously* substituting all upper (lower) bounds of α-cuts of fuzzy parameters in the technology matrix. The work attempted to provide a formal "rigorous" proof of the phenomena noticed earlier by Tan (2008), which examined the extent to which a final inventory was influenced by the vagueness of select parameters of the technology matrix. If true, the result would obviate the need to check the inventory at intermediate values of the α-cuts hypercube, in order to describe the full range of values a final inventory could assume. However, as discussed in the next section, the proof in HT is flawed. Even after taking the steps outlined in HT to achieve inverse positivity of the technology matrix at the core values, the inverse after simultaneously substituting all lower bounds, and a second one after substituting all upper bounds of the fuzzy system, would not necessarily address the full range of values of the final inventory, unless a stronger inverse-positivity condition on the system is satisfied.

2 The HT proof and underlying assumptions

As discussed in HT, a fuzzy number is a series of nested intervals around a most plausible value called the core. The support consists of the upper and lower bounds representing the vaguest specification of a parameter (see HT, Fig. 1). Narrower intervals (α-cuts) describe increasing certainty (plausibility) of fuzzy parameter value. Without loss of generality, we focus on the support of the fuzzy number in this analysis, *the 0-cut of the fuzzy number*. In the notation of Tan (2008), the triangular fuzzy number for the $(i,j)^{\text{th}}$

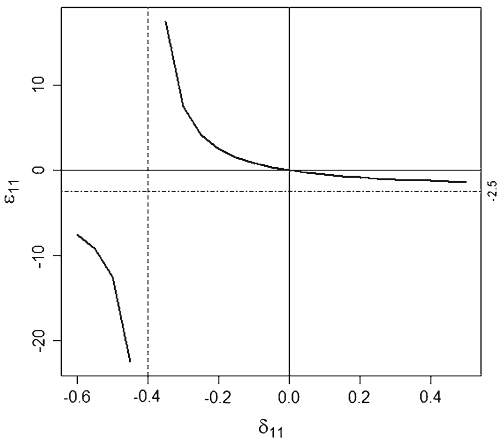

Fig. 1 Changes in $(A^{-1})_{11}$ of Tan's (2008) example 1 with respect to perturbations δ_{11} in a_{11}

element of a technology matrix A is $\left(a_{ij}+\delta_{ij}^{-}, a_{ij}, a_{ij}+\delta_{ij}^{+}\right)_T$ where, a_{ij} is the core value, and δ_{ij}^{-} (<0) and δ_{ij}^{+} (>0) determine the lower and upper endpoints of the support, respectively.

Note that by convention, process yields are represented by positive numbers in a technology matrix and inputs are represented as negative numbers. This above specification for a general fuzzy element suggests that the lower bound of an α-cut for an element in A corresponds to either the smallest output or maximum amount of input of that product; similarly, upper bounds of an α-cut correspond to the largest yield or least amounts of input of that product.

The proof in HT uses the famous Sherman–Morrison formula (Sherman and Morrison 1950) for the inverse of the perturbed matrix, A', corresponding to a perturbation δ_{ij} in just the (i,j)th element of the matrix A, which expresses a generic (k,l)th element of $(A')^{-1}$ as a function of δ_{ij} and some elements of the original matrix A^{-1}, i.e.,

$$\left((A')^{-1}\right)_{kl} = \left(A^{-1}\right)_{kl} + \varepsilon_{kl}, \qquad \text{(HT13)}$$

with the change ε_{kl} given by

$$\varepsilon_{kl} = -\frac{(A^{-1})_{ki}(A^{-1})_{jl}\delta_{ij}}{1+(A^{-1})_{ji}\delta_{ij}}. \qquad \text{(HT14)}$$

HT assumes that the core technology matrix A is inverse positive (i.e., $(A^{-1})_{kl}>0$ for all (k,l)), and then correctly argue that for a *positive perturbation*, it is easily seen that ε_{kl} is *negative* for all (k,l), and that it is a monotonically decreasing function of δ_{ij}. The negative change implies a reduction in the scaling factor associated with the kth process, when one unit of the lth product is desired.

However, for a *negative perturbation*, the proof simply asserts that "the argument also holds true if we look at the lower value of a_{ij} …in which case we find $\varepsilon_{kl}>0$" (HT, p. 1017). This assertion is not necessarily valid, in that ε_{kl} *is not always positive for a negative* δ_{ij}, since the sign of the denominator in the right hand side of (HT14), in fact, depends on the value of δ_{ij}. Note that the denominator is positive for all $\delta_{ij} > -1/(A^{-1})_{ji}$, and it is negative for all $\delta_{ij} < -1/(A^{-1})_{ji}$. Furthermore, as $\delta_{ij} \to -1/(A^{-1})_{ji}$, the denominator goes to zero, consequently, $\varepsilon_{kl} \to \pm\infty$. Thus, in general, a lower endpoint for the support of just one fuzzy element in the matrix could possibly violate the posited outcome of an inventory resulting in a higher number than at the core. We describe this phenomenon in the counterexample given below. In addition, it is worth noting that as the uncertainty in the upper bound of a fuzzy specification increases, the limiting value of ε_{kl} in (HT14) is given by

$$\lim_{\delta_{ij}\to\infty} \varepsilon_{kl} = -\left[\left(A^{-1}\right)_{ki}\left(A^{-1}\right)_{jl}\right]/\left(A^{-1}\right)_{ji}. \qquad (1)$$

Therefore, $\lim_{\delta_{ii}\to\infty} \varepsilon_{ii} = -(A^{-1})_{ii}$, and it follows from (HT13) that $\lim_{\delta_{ii}\to\infty} \left((A')^{-1}\right)_{ii} = 0$.

Of course, the inverse of any perturbed matrix is obtainable by iterative applications of the Sherman and Morrison formula to element-wise perturbations. However a valid iterative proof requires that the successive single-element perturbed technology matrices (A', A'',…) remain inverse positive at each step of the iteration. This is an implicit assumption in the proof, but not explicitly stated in the main theorem, and it may not always hold, as shown below. The point to note is that by specifying the fuzzy support for every element independently of others, and starting with an inverse positive core matrix, one cannot expect to preserve inverse positivity at each step of the sequential perturbation process.

Counterexample Tan's (2008) paper included several examples of technology matrix with fuzzy specification. In particular, we consider their example 1 of a two-product, two-process system for which uncertainties in the yields are described by the fuzzy specification

$$A = \begin{bmatrix} (0.5, 0.6, 1)_T & -1 \\ -1 & (3, 5, 6)_T \end{bmatrix} \qquad (2)$$

(i.e., $\delta_{11}^{-} = -0.1,\ \delta_{11}^{+} = 0.4,\ \delta_{22}^{-} = -2,\ \delta_{22}^{+} = 1$). The inverse of the matrix A at the core values is given by

$$A_0^{-1} = \begin{bmatrix} 2.5 & 0.5 \\ 0.5 & 0.3 \end{bmatrix}. \qquad (3)$$

Within the given range of fuzzy values in Eq. (2) above, the lower (upper) bounds indeed yield the largest (smallest) scale factors, and the desired result holds.

However, on examining the effect of δ_{11} over a slightly wider support set, we find that inverse positivity may be lost. Figure 1 describes the changes in the (1,1)th element of

A'^{-1} with respect to a perturbation, δ_{11}, in the element a_{11}. Changes in each of the other entries of A'^{-1} exhibit similar behavior, i.e., they have the same vertical asymptote with the scale of each determined by Eq. (HT 14), and the horizontal limiting value determined by Eq. (1). Note that as δ_{11} approaches $-1/\left(A_0^{-1}\right)_{11} = -0.4$, the matrix A' becomes nearly singular and the changes to each inverse element become drastic. For, $-0.6 < \delta_{11} < -0.4$, the first process continues to make small amounts of the first product with a given input, yet every scale factor would be negative.

Now, a simple adjustment of δ_{11}^{-} in the above example demonstrates that a complete reversal in the sign of scaling factors is possible when simultaneously substituting lower bounds. If the lower endpoint of the support on the a_{11} element was changed to 0.3 (i.e., $\delta_{11}^{-} = -0.3$ which is still greater than −0.4, the value at which the vertical asymptote is achieved), then after perturbing just the (1,1) element in the core matrix to 0.3, $(A')^{-1}$ remains positive. However, after substituting lower bounds on the two outputs, in Tan's notation, the inverse of the matrix

$A_{L,0} = \begin{bmatrix} 0.3 & -1 \\ -1 & 3 \end{bmatrix}$ turns out to be $\left(A_{L,0}\right)^{-1} = \begin{bmatrix} -30 & -10 \\ -10 & -3 \end{bmatrix}$,

whose elements are all negative. The inventories derived from the lower bounds would no longer be the largest of all possible values. If the support set for the fuzzy outputs of the system were established independently for each element, presumably reflecting some knowledge about the potential yield of each process, the resulting matrix inverse can contain strictly negative elements, violating the inverse-positivity outcome.

3 Sufficient conditions for HT result to hold

The counterexample in the previous section demonstrated that for a combination of negative perturbations to two or more core values of fuzzy technology matrix, the inverse-positivity condition, implicitly assumed in HT, may be violated. However, identification of minimum and maximum scale factors (and consequently, inventories) is still possible with a careful statement of inverse-positivity condition.

The steps to guarantee inverse positivity impose an additional structure on the technology matrix. After redefinition, no two processes make the same product; after allocation, no two products come from the same process.

Even though HT asserted that there was no need to make the matrix “symmetric” in the sense that the ith row product is produced by the ith column process, this step simply amounts to permuting equations, leaving the system of equations unchanged. By so doing, A is a square matrix with strictly positive diagonal elements ($a_{ii}>0$) and non-positive off-diagonal elements ($a_{ij}\leq 0$, $i\neq j$) If this matrix is also inverse positive, it is known to be an M-matrix (Fan 1958). The class of M-matrices is known to have many interesting properties (Berman and Plemmons 1979). In extending Ostrowski's results (Ostrowski 1937) on the inverse of a M-matrix, Fan (Fan 1960) recognized that when both A and B were M-matrices, with $A\leq B$, *in the sense that inequalities hold element-wise*, as one element is perturbed at a time in A to reach B, the chain of matrices in between were also M-matrices with decreasing scale factors, and proved the following result:

Proposition 1 (Fan 1960, p. 43) If A and B are non-singular M-matrices, with $A\leq B$ then $A^{-1}\geq B^{-1}$.

It is worth noting the proof in HT for positive perturbations in each element of A, one at a time to reach B, is almost exactly the same as Fan's classical proof. But Fan's proof also shows that the inverse-positive property is preserved at each step. When $A\leq B$, it may be worthwhile to call A as a less efficient system than B; since A is the technology matrix in which all outputs are smaller than the corresponding outputs in B and all inputs are larger than the corresponding inputs in B. When both A and B are invertible M-matrices, proposition 1 guarantees that each vector of scale factors (same column of A^{-1} and B^{-1}) has the desired monotonically decreasing property. Noting that lower bounds on fuzzy inputs actually denote most use of that input, while the upper bounds describe least use of the input, the technically correct statement of the sufficient conditions for fuzzy error propagation in matrix-based LCI is as follows:

Proposition 2 If the least efficient system and the most efficient system at any given α-cut are M-matrices (i.e., inverse positive), then the largest scale factors will be associated with the least efficient system, and the smallest scale factors will be associated with the most efficient system.

This makes intuitive sense, and it is the analogue of Buckley's theorem 1 (Buckley 1989), for fuzzy input-output analysis. By shifting emphasis from inverse positivity of the core matrix to inverse positivity of the matrices corresponding to all lower (all upper) bounds of any α-cut, the minimum and maximum inventory values can be assessed at the most and least efficient specifications, respectively.

4 Conclusions

In conclusion, the proof given by HT is flawed. A careful examination of Eq. (HT 14) shows that the specification of lower bounds of a support (or any α-cut) of a fuzzy number could violate the desired properties. However, if the matrices corresponding to the least efficient (all lower bounds) and the most efficient (all upper bounds) systems are inverse positive,

the range of the inventories could be obtained by the evaluations at the two extremes corresponding to all lower and all upper bounds.

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